Fast Fourier Transform

The fast Fourier transform (FFT) is an algorithm for computing the discrete Fourier transform <u>very</u> quickly.

- developed in the 1960's
- used for mp3, jpeg, ...

→ On a 2.5 GHz processor (with one thread) 5 seconds of a 44 kHz audio sample requires ~ <u>13 hours</u> of computation w/DFT but ~ <u>4 seconds</u> with FFT (< 1 second with more cores)

FFT works by writing the Fourier transf. of a big vector as a combination of Fourier transforms of smaller vectors.

$$\underline{f} = (\underline{f_0}, \underline{f_1}, \underline{f_2}, \underline{f_3}, \underline{f_4}, \underline{f_5}, \underline{f_6}, \dots)$$

$$\boxed{f_e = (f_0, f_2, f_4, \dots)}$$
"even" subvector

$$\boxed{f_o = (f_1, f_3, f_5, \dots)}$$
"odd" subvector

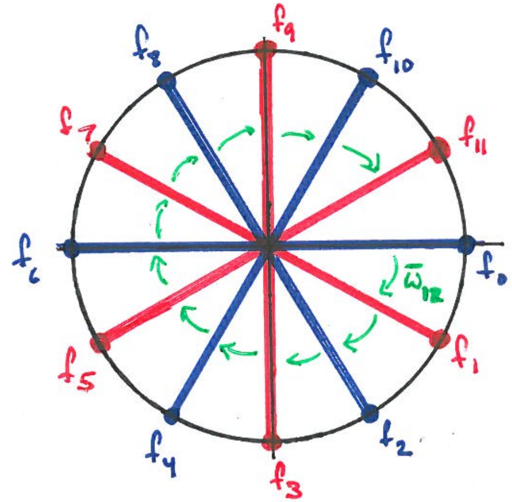
→ <u>Note</u>: length of $\underline{f}$ <u>must</u> be a multiple of 2

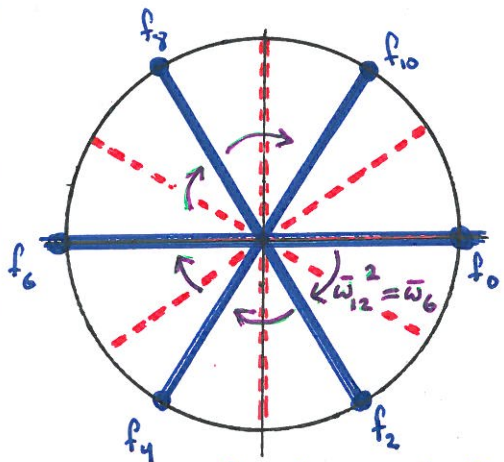
We can combine multiple reductions if

$$\text{length}(\underline{f}) = 2^p \quad \text{(power of 2)}$$

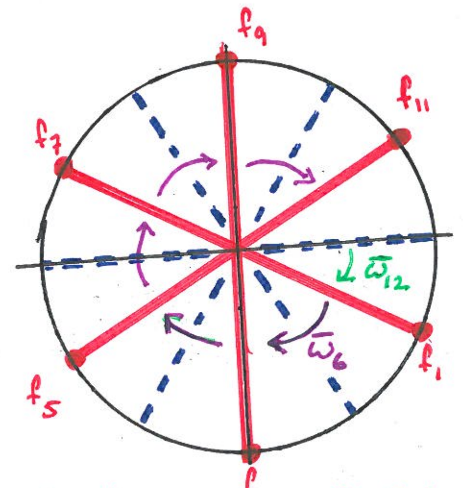
↳ Serious FFT implimentations <u>require</u> length $= 2^p$

You can see the basic idea in computation of the first Fourier coefficient, $\underline{c}_1$:



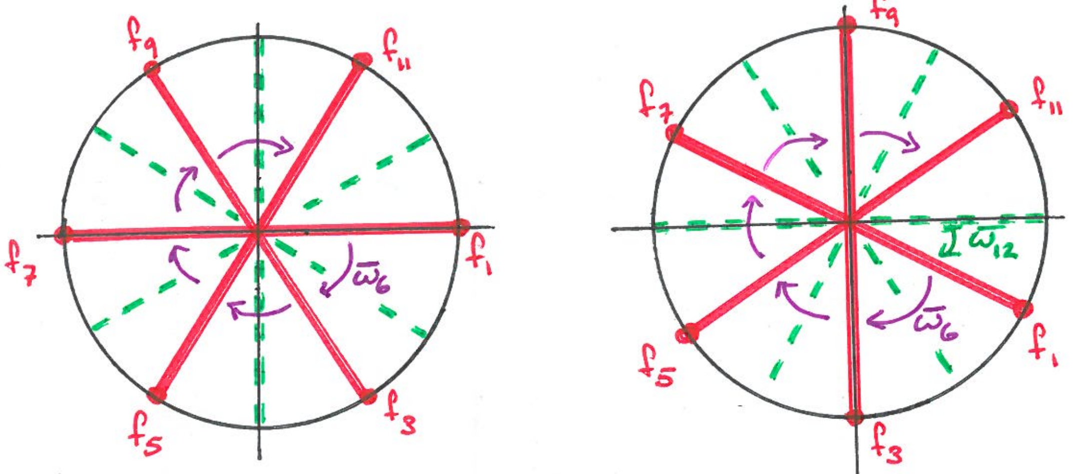


$f_e = (f_0, f_2, f_4, f_6, f_8, f_{10})$



$f_o = (f_1, f_3, f_5, f_7, f_9, f_{11})$

We can recover the c_1 coefficient of $\underline{f}$ by adding the c_1 coefficient of $\underline{f}_e$ to a <u>rotated</u> c_1 coefficient of $\underline{f}_o$



$\mathcal{F}_1\{\underline{f}_o\}$ $\xrightarrow{\text{rotate}}$ $\bar{\omega}_{12}\,\mathcal{F}_1\{\underline{f}_o\}$

& $\underline{f}_o = (f_1, f_3, f_5, f_7, f_9, f_{11})$

"<u>odd</u>" subvector of $\underline{f}$. ($\underline{f}_o$ = "f_{odd}")

General Formula:

$$\mathcal{F}_k\{f\} = \frac{1}{2}\left(\mathcal{F}_k\{\underline{f}_e\} + \bar{\omega}_N^k\,\mathcal{F}_k\{\underline{f}_o\}\right) \quad \left(k < \frac{N}{2}\right)$$

for the remaining coefficients

$$\mathcal{F}_{N/2+k}\{f\} = \frac{1}{2}\left(\mathcal{F}_k\{f_e\} - \bar{\omega}_N^k\,\mathcal{F}_k\{f_o\}\right)$$

$$\left(\bar{\omega}_N^{N/2+k} = \bar{\omega}_N^{N/2}\,\bar{\omega}_N^{k} = -\bar{\omega}_N^{k}\right.$$

(1/2 is because Fourier transform calculates <u>averages</u>)

<u>EX</u>: Use FFT to compute $\mathcal{F}\{(2,-1,3,-2)\}$

$f_e = (2, 3)$ $\qquad$ $f_o = (-1, -2)$

$\mathcal{F}_0\{f_e\} = \frac{1}{2}(2+3) = 5/2$ $\qquad$ $\mathcal{F}_0\{f_o\} = \frac{1}{2}(-1-2) = -3/2$

$\mathcal{F}_1\{f_e\} = \frac{1}{2}(2-3) = -1/2$ $\qquad$ $\mathcal{F}_1\{f_o\} = 1/2(-1+2) = 1/2$

(<u>Note</u>: $\bar{\omega}_4 = -i$)

(k)

$\mathcal{F}_0\{f\} = \frac{1}{2}\left(5/2 - 3/2\right) = 1/2$

$\mathcal{F}_1\{f\} = \frac{1}{2}\left(-1/2 - 1/2\, i\right) = 1/4(-1-i)$

$\left(\frac{N}{2}+k\right)$

$\mathcal{F}_2\{f\} = \frac{1}{2}\left(5/2 + 3/2\right) = 2$

$\mathcal{F}_3\{f\} = 1/2\left(-1/2 + 1/2\, i\right) = 1/4(-1+i)$

Inverse fast Fourier is the same, except it uses ω_N instead of $\bar{\omega}_N$ and does <u>not</u> avg

$$\mathcal{F}_n^{-1}\{\underline{c}\} = \mathcal{F}_n^{-1}\{\underline{c}_e\} + \omega_N^n\,\mathcal{F}_n^{-1}\{\underline{c}_o\}$$

$$\mathcal{F}_{N/2+n}^{-1}\{\underline{c}\} = \mathcal{F}_n^{-1}\{\underline{c}_e\} - \omega_N^n\,\mathcal{F}_n^{-1}\{c_o\}$$

(<u>Note</u>: No $\frac{1}{2}$.)

EX Use IFFT to compute

$$\mathcal{F}^{-1}\left\{\left(\tfrac{1}{2},\ \tfrac{1}{4}(-1-i),\ 2,\ \tfrac{1}{4}(-1+i)\right)\right\}$$

$c_e = \left(\tfrac{1}{2},\ 2\right)$ $\qquad$ $c_o = \left(\tfrac{1}{4}(-1-i),\ \tfrac{1}{4}(-1+i)\right)$

$\mathcal{F}_0^{-1}\{c_e\} = \tfrac{1}{2} + 2 = \tfrac{5}{2}$ $\qquad$ $\mathcal{F}_0^{-1}\{c_o\} = \tfrac{1}{4}(-1-i) + \tfrac{1}{4}(-1+i) = -\tfrac{1}{2}$

$\mathcal{F}_1^{-1}\{c_e\} = \tfrac{1}{2} - 2 = -\tfrac{3}{2}$ $\qquad$ $\mathcal{F}_1^{-1}\{c_o\} = \tfrac{1}{4}(-1-i) - \tfrac{1}{4}(-1+i) = -\tfrac{i}{2}$

$$\mathcal{F}_0^{-1}\{\underline{c}\} = \tfrac{5}{2} - \tfrac{1}{2} = 2$$
$$\mathcal{F}_1^{-1}\{\underline{c}\} = -\tfrac{3}{2} - \tfrac{i}{2}\cdot i = -1$$
$$\mathcal{F}_2^{-1}\{\underline{c}\} = \tfrac{5}{2} + \tfrac{1}{2} = 3$$
$$\mathcal{F}_3^{-1}\{\underline{c}\} = -\tfrac{3}{2} + \tfrac{i}{2}\cdot i = -2$$

EX Use FFT to compute

$$\mathcal{F}\{(2, -1, 3, -2, -4, -3, 2, 1)\}$$

$f_e = (2, 3, -4, 2)$ $\qquad$ $f_o = (-1, -2, -3, 1)$

(Note: $\bar{\omega}_8 = \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i$)

$$\mathcal{F}_0\{f_e\} = \tfrac{1}{4}(2+3-4+2) = \tfrac{3}{4}$$
$$\mathcal{F}_1\{f_e\} = \tfrac{1}{4}(2-3i+4+2i) = \tfrac{1}{4}(6-i)$$
$$\mathcal{F}_2\{f_e\} = \tfrac{1}{4}(2-3-4-2) = -\tfrac{7}{4}$$
$$\mathcal{F}_3\{f_e\} = \overline{\mathcal{F}_1\{f_e\}} = \tfrac{1}{4}(6+i) \quad \text{(short-cut)}$$

(EX continued)

$$\mathcal{F}_0\{f_o\} = \tfrac{1}{4}(-1-2-3+1) = -\tfrac{5}{4}$$
$$\mathcal{F}_1\{f_o\} = \tfrac{1}{4}(-1+2i+3+i) = \tfrac{1}{4}(2+3i)$$
$$\mathcal{F}_2\{f_o\} = \tfrac{1}{4}(-1+2-3-1) = -\tfrac{3}{4}$$
$$\mathcal{F}_3\{f_o\} = \overline{\mathcal{F}_1\{f_o\}} = \tfrac{1}{4}(2-3i) \quad \text{(short-cut)}$$

$$\mathcal{F}_0\{f\} = \tfrac{1}{2}\left(\tfrac{3}{4} - \tfrac{5}{4}\right) = -\tfrac{1}{4}$$
$$\mathcal{F}_1\{f\} = \tfrac{1}{2}\left(\tfrac{1}{4}(6-i) + \tfrac{1}{4}(2+3i)\left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i\right)\right)$$
$$= \tfrac{1}{8}\left(\left(6 + \tfrac{5\sqrt{2}}{2}\right) + \left(-1 + \tfrac{\sqrt{2}}{2}\right) i\right)$$
$$\mathcal{F}_2\{f\} = \tfrac{1}{2}\left(-\tfrac{7}{4} + \left(-\tfrac{3}{4}\right)(-i)\right)$$
$$= \tfrac{1}{8}(-7 + 3i)$$
$$\mathcal{F}_3\{f\} = \tfrac{1}{2}\left(\tfrac{1}{4}(6+i) + \tfrac{1}{4}(2-3i)\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} i\right)\right)$$
$$= \tfrac{1}{8}\left(\left(6 - \tfrac{5\sqrt{2}}{2}\right) + \left(1 + \tfrac{\sqrt{2}}{2}\right) i\right)$$
$$\mathcal{F}_4\{f\} = \tfrac{1}{2}\left(\tfrac{3}{4} + \tfrac{5}{4}\right) = 1$$

The rest are conjugate.

$$\mathcal{F}_5\{f\} = \overline{\mathcal{F}_{8-5}\{f\}} = \overline{\mathcal{F}_3\{f\}}$$
$$= \tfrac{1}{8}\left(\left(6 - \tfrac{5\sqrt{2}}{2}\right) - \left(1 + \tfrac{\sqrt{2}}{2}\right) i\right)$$
$$\mathcal{F}_6\{f\} = \overline{\mathcal{F}_2\{f\}}$$
$$= \tfrac{1}{8}(-7 - 3i)$$
$$\mathcal{F}_7\{f\} = \overline{\mathcal{F}_1\{f\}}$$
$$= \tfrac{1}{8}\left(\left(6 + \tfrac{5\sqrt{2}}{2}\right) - \left(-1 + \tfrac{\sqrt{2}}{2}\right) i\right)$$

Summary:

FFT formula:

$$f \to f_{even} = f_e, \quad f \to f_{odd} = f_o$$

(first half of c) $\mathcal{F}_k\{f\} = \frac{1}{2}\left(\mathcal{F}_k\{f_e\} + \bar{\omega}_N^k \mathcal{F}_k\{f_o\}\right)$

(second half) $\mathcal{F}_{N/2+k}\{f\} = \frac{1}{2}\left(\mathcal{F}_k\{f_e\} - \bar{\omega}_N^k \mathcal{F}_k\{f_o\}\right)$

IFFT formula:

$$c \to c_{even} = c_e, \quad c \to c_{odd} = c_o$$

(first half of f) $\mathcal{F}_n^{-1}\{c\} = \mathcal{F}_n^{-1}\{c_e\} + \omega_N^n \mathcal{F}_n^{-1}\{c_o\}$

(second half) $\mathcal{F}_{N/2+n}^{-1}\{c\} = \mathcal{F}_n^{-1}\{c_e\} - \omega_N^n \mathcal{F}_n^{-1}\{c_o\}$